AN UNCERTAINTY PRINCIPLE AND SAMPLING INEQUALITIES IN BESOV SPACES

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Abstract. We extend Strichartz's uncertainty principle [21] from the setting of the Sobolev space $W^{1,2}(\mathbb{R})$ to more general Besov spaces $B^{1/p}_{p,1}(\mathbb{R})$. The main result gives an estimate from below of the trace of a function from the Besov space on a uniformly distributed discrete subset. We also prove the corresponding result in the multivariate case and discuss some applications to irregular approximate sampling in critical Besov spaces.

1. Introduction

1.1. **Motivation and main results.** Let us recall the classical Heisenberg uncertainty principle

$$\int_{\mathbb{R}} |\zeta|^2 |\hat{f}(\zeta)|^2 \,\mathrm{d}\zeta \int_{\mathbb{R}} |x|^2 |f(x)|^2 \,\mathrm{d}x \geq \frac{\|f\|_{L^2}^4}{16\pi^2},$$

where $f \in L^2(\mathbb{R})$ and $\hat{f}(\zeta) = \displaystyle\int_{\mathbb{R}} f(x)\exp(-2\pi i x\zeta)\,\mathrm{d}x$. It is frequently refereed to as a restriction for a simultaneous good localization of a function and its Fourier transform around the origin. It can also be read as an inequality preventing smooth functions with bounded norm (in some homogeneous Sobolev space) to be well concentrated, since

$$\int_{\mathbb{R}} |\zeta|^2 |\hat{f}(\zeta)|^2 d\zeta = \|f'\|_{L^2}^2 = \|f\|_{\dot{W}^{1,2}}^2.$$

Here we use standard notation, $W^{s,p}(\mathbb{R}) = \{f \in L^p : (I-\Delta)^{s/2} f \in L^p\}$ for $s \geq 0$ and $p \geq 1$ and define the (homogeneous) norm in $W^{s,p}(\mathbb{R})$ by $\|f\|_{\dot{W}^{s,p}} = \|\Delta^{s/2} f\|_{L^p}$ for $f \in W^{s,p}(\mathbb{R})$.

The Heisenberg inequality was further generalized by Cowling and Price:

Theorem (Cowling-Price, [4]). *Let $p, q \in [1,\infty]$ and $a, b > 0$. There exists a constant K such that*

$$\big\||x|^a f\big\|_{L^p} + \big\||\xi|^b \widehat{f}\big\|_{L^q} \geq K\|f\|_{L^2} \tag{1.1}$$

for all $f \in L^2(\mathbb{R})$ if and only if

$$a > \frac{1}{2} - \frac{1}{p} \quad \text{and} \quad b > \frac{1}{2} - \frac{1}{q}.$$

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If this is the case, let γ be given by

$$\gamma\left(a-\frac{1}{2}+\frac{1}{p}\right)=(1-\gamma)\left(b-\frac{1}{2}+\frac{1}{q}\right)$$

then there exists a constant K such that

$$\big\||x|^a f\big\|_{L^p}^{\gamma}\big\||\xi|^b\widehat{f}\big\|_{L^q}^{1-\gamma}\geq K\|f\|_{L^2}.$$

Again, when $q=2$, $\big\||\xi|^b\widehat{f}\big\|_{L^2(\mathbb{R})}=\|f\|_{\dot{W}^{b,2}}$.

In [21] R. Strichartz obtained a number of uncertainty inequalities in Euclidean spaces, connecting smoothness of a function to its concentration on a suitably uniformly distributed set. His starting point was the following:

Theorem (Strichartz, [21]**).** *Let $\{a_j\}_{-\infty}^{\infty}$ be a sequence of points with $a_{j+1}-a_j\leq b$, $\lim_{j\to\pm\infty}a_j=\pm\infty$, and let $f\in W^{1,2}(\mathbb{R})$. If $\sum_j|f(a_j)|^2\leq(1-\varepsilon)^2b^{-1}\|f\|_{L^2}^2$ then $\|f'\|_{L^2}^2\geq c\varepsilon^2b^{-2}\|f\|_{L^2}^2$.*

Our aim here is to extend this result to L^p-smoothness setting as the Cowling-Price Theorem extends the classical Heisenberg Inequality. To be able to consider the values of a functions f at some points, we assume that the function possesses some smoothness. For the classical case of L^2-norm we need at least derivative of order $1/2$. The embedding theorem suggests that we would require at least $1/p$-smoothness for L^p-norms. The right scale turns out to be that of Besov spaces. Our main result gives bounds (also from below) on the trace of functions from Besov spaces on some uniformly distributed union of subspaces. Note also that Paley-Wiener spaces are included in Besov spaces so that our results cover some results in sampling theory, though with non-optimal density. In particular, it implies the following statement.

Theorem 1.1. *Assume that m is an integer $0<m\leq d$, and $1\leq p<\infty$, let also D be given. There exist constants δ,C_1,C_2 that depend only on d,p,m,D such that if $f\in B_{p,1}^{m/p}(\mathbb{R}^d)$, $\|f\|_{\dot{B}_{p,1}^{m/p}}\leq N\|f\|_p$, and $b^{m/p}<\delta N^{-1}$ then for any $G=\mathbb{R}^{m-d}\times E$, where $E\subset\mathbb{R}^m$ is a discrete set such that $\cup_{x\in E}[x-b,x+b]^m=\mathbb{R}^m$ and each point belongs to at most D distinct sets in this union, the following inequality holds*

$$C_1\|f\|_{L^p}\leq b^{m/p}\left(\int_G|f(x)|^p\,d\mathcal{H}^{d-m}(x)\right)^{1/p}\leq C_2\|f\|_{L^p}.$$

We remark that the right hand side inequality follows from localization and trace properties of the Besov spaces, the aim is to prove the left hand side estimate. However our argument gives both inequalities simultaneously and in particular provides a quite simple proof of the classical trace estimate. We consider the critical Besov spaces for which the trace inequality holds, the result does not hold in the spaces $B_{p,q}^s$ with $s<m/p$. The concentration principle says that a function satisfying the conditions above cannot have too many gaps, the precise statement is given by the left hand side inequality above. We also rewrite the result as sampling inequalities for functions in the corresponding Besov spaces and apply it to give estimates for irregular approximate sampling. Similar one-dimensional approximation results for the case of regular samples were obtained recently in [12].

The remaining of the paper is organized as follows. We start by recalling the main facts on Besov spaces that we use. Section 2 is then devoted to the one-dimensional version of Theorem 1.1 and some corollaries. In section 3 we prove the main result in higher dimensions and apply it to sampling theory.

1.2. **Preliminaries on wavelets and Besov spaces.** We recall very briefly the basics of multiresolution wavelet analysis (for details see for instance [5]). For an arbitrary integer $N \geq 1$ one can construct functions $\psi^0 \in \mathcal{C}^N(\mathbb{R})$ (called the scaling function) and $\psi^1 \in \mathcal{C}^N(\mathbb{R})$ (called the mother wavelet), with

(i) $\psi^0, \psi^1 \in \mathcal{C}^N(\mathbb{R})$ and are real valued;
(ii) for $\ell = 0, 1$, $m = 0, \ldots, N$ and $k \geq 1$ there exists C_k such that $|\partial^m \psi^\ell(x)| \leq C_k(1+|x|)^{-k}$;
(iii) the set of functions $\psi^1_{j,k} : x \to 2^{j/2}\psi^1(2^j x - k)$, $j, k \in \mathbb{Z}$ forms an orthonormal basis for $L^2(\mathbb{R})$.
(iv) ψ^1 has N vanishing moments $\displaystyle\int_{\mathbb{R}} x^k \psi^1(x)\,\mathrm{d}x = 0$ for $k = 0, \ldots, N-1$.

We will then say that ψ^1 is N-regular.

Now, in higher dimension $d \geq 2$, we introduce

$$\mathbf{0}^d = (0, \ldots, 0), \quad \mathbf{1}^d = (1, \ldots, 1) \quad \text{and} \quad L^d = \{0,1\}^d \setminus \{\mathbf{0}^d\}.$$

An orthonormal basis of $L^2(\mathbb{R}^d)$ is then obtained by tensorization: for $\lambda = (\mathbf{k}, \mathbf{l}) \in \mathbb{Z}^d \times \{0,1\}^d$ with $\mathbf{k} = (k_1, \ldots, k_d)$ and $\mathbf{l} = (l_1, \ldots, l_d)$ we define

$$\psi_{j,\lambda}(x) = \prod_{i=1}^d \psi^{l_j}_{j,k_i}(x).$$

Any function $f \in L^2(\mathbb{R}^d)$ can then be written as

$$f = \sum_{(j,\lambda) \in \mathbb{Z} \times \mathbb{Z}^d \times L^d} c_{j,\lambda}\psi_{j,\lambda} \quad \text{with } c_{j,\lambda} = \int_{\mathbb{R}^d} f(x)\psi_{j,\lambda}(x)\,\mathrm{d}x.$$

Let $s > 0$, $1 \leq p; q \leq \infty$. Assume that ψ^1 is at least $[s]+1$-regular. According to [3, 13, 17], the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^d)$ can be defined as the space of all locally integrable functions such that the Besov norm

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^d)} = \left(\sum_{j \in \mathbb{Z}} \left[2^{(s-d/p+d/2)j}\left(\sum_{\lambda \in \mathbb{Z}^d \times L^d} |c_{j,\lambda}|^p\right)^{\frac{1}{p}}\right]^q\right)^{\frac{1}{q}} \tag{1.2}$$

is finite.

An alternative definition is as follows [18, 22, 23]. Fix an arbitrary non-negative smooth function ρ supported in $\{y \in \mathbb{R}^d : \dfrac{1}{2} < |y| < 2\}$ such that, for $y \neq 0$,

$$\sum_{j \in \mathbb{Z}} \rho(2^{-j}y) = 1.$$

Denote by $\mathcal{F}$ the Fourier transform an $ß'(\mathbb{R}^d)$ (the space of tempered distributions) and by $\mathcal{F}^{-1}$ the inverse Fourier transform. Then $\dot{B}^s_{p,q}(\mathbb{R}^d)$ is the space of all tempered distributions such that

$$\left(\sum_{j \in \mathbb{Z}} 2^{jsq}\big\|\mathcal{F}^{-1}\left[\rho(2^{-j}\cdot)\mathcal{F}[f]\right]\big\|^q_{L^p(\mathbb{R}^d)}\right)^{\frac{1}{q}}. \tag{1.3}$$

Moreover, this quantity defines an equivalent norm to $\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^d)}$. Using this norm, we see that there is a constant C such that, if $\operatorname{supp}\mathcal{F}(f) \cap \{|\xi| < b\} = \emptyset$ then $\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^d)} \geq$

$Cb^{s-s'}\|f\|_{\dot{B}^{s'}_{p,q}(\mathbb{R}^d)}$ if $s>s'$. On the other hand, if we write $PW^p_b=\{f\in L^p\,:\operatorname{supp}\mathcal{F}(f)\subset[-b,b]\}$ for the closed subspace of L^p with distributional Fourier transform supported in $[-b,b]$, then $PW^p_b\subset B^s_{p,q}$ for every $s>0$ and $q\geq 1$.

Note that Besov spaces are also related to Sobolev spaces in the following way for $p\geq 1$ and $\varepsilon>0$, $q'\geq q>0$

$$W^{s+\varepsilon,p}\hookrightarrow\dot{B}^s_{p,q}\hookrightarrow\dot{B}^s_{p,q'}\hookrightarrow W^{s-\varepsilon,p}.$$

Functions in the Besov space $B^{d/p}_{p,1}(\mathbb{R}^d)$ coincide with continuous ones almost everywhere. We will take the continuous representative. The trace of a function in $B^{d/p}_{p,1}(\mathbb{R}^d)$ on a subspace M of codimension r belongs to $B^{r/p}_{p,1}(M)$. We refer the reader to [23] and references there for the details; interesting results on local regularity of functions from the critical Besov spaces can be found in [11].

2. Sampling and uncertainty in dimension one

2.1. **Localization inequalities.** First, we prove one-dimensional version of Theorem 1.1 to demonstrate the main ideas avoiding technical complications. In the next section we outline the changes that should be done in multivariate case. The proof is rather classical and shares some features with Stricharz's original approach but employs wavelet decomposition. For applications of similar techniques to irregular sampling see e.g. [8, 9].

Proposition 2.1. *Assume that $f\in B^{1/p}_{p,1}(\mathbb{R})$, for some $p\geq 1$. If $\{a_n\}_{n\in\mathbb{Z}}$ is an increasing sequence, $\lim_{\pm\infty}a_n=\pm\infty$, $a_{n+1}-a_n\leq b$ and*

$$\sum_{n\in\mathbb{Z}}|f(a_n)|^p\leq(1-\varepsilon)^p b^{-1}\|f\|^p_{L^p},$$

then $\|f\|_{\dot{B}^{1/p}_{p,1}}\geq c\varepsilon b^{-1/p}\|f\|_{L^p}$, where c depends on p only.

Moreover, there is a constant C, depending only on p, such that, if $b\leq C\varepsilon\left(\dfrac{\|f\|_{L^p}}{\|f\|_{\dot{B}^{1/p}_{p,1}}}\right)^p$ and a_j are as above with $b/2\leq a_{j+1}-a_j\leq b$ then

$$\frac{1}{2b^{1/p}}\|f\|_{L^p}\leq\left(\sum_{n\in\mathbb{Z}}|f(a_n)|^p\right)^{1/p}\leq\frac{5}{2b^{1/p}}\|f\|_{L^p}.$$

Proof. Let ψ be a wavelet function such that $\psi(x)=0$ when $|x|>R$ ($\psi=\psi^1$ from the previous section). Write

$$f=\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\langle f,\psi_{j,k}\rangle\psi_{j,k},$$

where $\psi_{j,k}$ are defined as above. It follows from the estimates below that the series converges uniformly. For each $n\in\mathbb{Z}$, let $I_n=[(a_{n-1}+a_n)/2,(a_n+a_{n+1})/2]$, then $b_n=|I_n|\leq b$. Fix some n and let $x\in I_n$. Then

$$\begin{aligned}f(x)-f(a_n)&=\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\langle f,\psi_{j,k}\rangle\big(\psi_{j,k}(x)-\psi_{j,k}(a_n)\big)\\&=\sum_{j\in\mathbb{Z}}2^{j/2}\sum_{k\in\mathbb{Z}}\langle f,\psi_{j,k}\rangle\big(\psi_0(2^jx-k)-\psi_0(2^ja_n-k)\big).\end{aligned}$$

If $\psi(2^jx-k)\neq 0$ then

$$2^jx-R\leq k\leq 2^jx+R.$$

Write

$$Z_x(j)=\{k\,:2^jx-R\le k\le 2^jx+R\},\ Z_x(n,j)=Z_x(j)\cup Z_{a_n}(j),\ Z(n,j)=\bigcup_{x\in I_n}Z_x(j),$$

and note that $|Z_x(j)|\le 2R+1$, therefore $|Z_x(n,j)|\le 4R+2$.

Now,

$$\begin{aligned}(2.4)\quad |f(x)-f(a_n)|&\le\sum_{j\in\mathbb{Z}}2^{j/2}\max_k|\psi(2^jx-k)-\psi(2^ja_n-k)|\sum_{k\in\mathbb{Z}_x(n,j)}|\langle f,\psi_{j,k}\rangle|\\&\le(4R+2)^{1/p'}\sum_{j\in\mathbb{Z}}2^{j/2}\max_k|\psi(2^jx-k)-\psi(2^ja_n-k)|E_{n,j},\end{aligned}$$

the last inequality follows from Hölder's inequality with the notation $\dfrac{1}{p}+\dfrac{1}{p'}=1$ and

$$E_{n,j}=\left(\sum_{k\in Z(n,j)}|\langle f,\psi_{j,k}\rangle|^p\right)^{1/p}. \tag{2.5}$$

Further, we obtain $|Z(n,j)|\le 2R+1+2^jb$ and if $|n-m|>2^{1-j}Rb^{-1}+1$ then $Z(n,j)\cap Z(m,j)=\emptyset$. Therefore, for each j, each $k\in\mathbb{Z}$ belongs to at most $2^{1-j}Rb^{-1}+2$ different $Z(n,j)$'s. We choose j_0 such that $2^{-j_0}\in(b,2b]$, and set

$$M_j:=\sup_{k\in\mathbb{Z}}|\{n\,:k\in Z(n,j)\}|.$$

Then we get

$$M_j\le 2^{1-j}Rb^{-1}+2\le\begin{cases}C & \text{if } j\ge j_0\\ Cb^{-1}2^{-j} & \text{if } j<j_0\end{cases}, \tag{2.6}$$

where C depends on R only.

Clearly, for $x,a\in I_n$ we have

$$|\psi(2^jx-k)-\psi(2^ja-k)|\le\begin{cases}2\|\psi\|_\infty,\\ 2^j|x-a|\|\psi'\|_\infty.\end{cases}$$

Then there exists a constant C that depends only on ψ such that

$$|f(x)-f(a_n)|\le C\sum_{j\ge j_0}2^{j/2}E_{n,j}+C\sum_{j<j_0}2^{3j/2}|x-a_n|E_{n,j}. \tag{2.7}$$

Taking the L^p-norms over $x\in I_n$ and applying the triangle inequality, we get

$$\begin{aligned}\left|\left(\int_{I_n}|f(x)|^p\,\mathrm{d}x\right)^{1/p}-|f(a_n)|b_n^{1/p}\right|&\le\left(\int_{I_n}|f(x)-f(a_n)|^p\,\mathrm{d}x\right)^{1/p}\\&\le C\sum_{j\ge j_0}2^{j/2}b^{1/p}E_{n,j}+C\sum_{j<j_0}2^{3j/2}b^{1+1/p}E_{n,j}.\end{aligned}$$

It remains to take the ℓ^p norm in n to obtain

$$\begin{aligned}\left|\|f\|_{L^p(\mathbb{R})}-\left(\sum_{n\in\mathbb{Z}}b_n|f(a_n)|^p\right)^{1/p}\right| &\leq \left(\sum_{n\in\mathbb{Z}}\left[\int_{I_n}|f(x)|^p\,\mathrm{d}x-|f(a_n)|^pb_n\right]\right)^{1/p}\\ &\leq C\sum_{j\geq j_0}2^{j/2}\left(\sum_{n\in\mathbb{Z}}\sum_{k\in Z(n,j)}|\langle f,\psi_{j,k}\rangle|^p\right)^{1/p}b^{1/p}\\ &\quad +C\sum_{j<j_0}2^{3j/2}\left(\sum_{n\in\mathbb{Z}}\sum_{k\in Z(n,j)}|\langle f,\psi_{j,k}\rangle|^p\right)^{1/p}b^{1+1/p}.\end{aligned}$$

We have also

$$\sum_{n\in\mathbb{Z}}\sum_{k\in Z(n,j)}|\langle f,\psi_{j,k}\rangle|^p\leq M_j\sum_{k\in\mathbb{Z}}|\langle f,\psi_{j,k}\rangle|^p,$$

and applying (2.6), we get

$$\begin{aligned}\left|\|f\|_{L^p(\mathbb{R})}-\left(\sum_{n\in\mathbb{Z}}b_n|f(a_n)|^p\right)^{1/p}\right| &\leq C\sum_{j\geq j_0}2^{j/2}\left(\sum_{k\in\mathbb{Z}}|\langle f,\psi_{j,k}\rangle|^p\right)^{1/p}b^{1/p}\\ &\quad +C\sum_{j<j_0}2^{(2-1/p)j}\left(\sum_{k\in\mathbb{Z}}|\langle f,\psi_{j,k}\rangle|^p\right)^{1/p}b.\end{aligned}$$

Finally, notice that if $j\leq j_0$ then $2^{(1-1/p)j}\leq Cb^{1/p-1}$, hence

$$\begin{aligned}\left|\|f\|_{L^p(\mathbb{R})}-\left(\sum_{n\in\mathbb{Z}}b_n|f(a_n)|^p\right)^{1/p}\right| &\leq C\sum_{j\in\mathbb{Z}}2^{j/2}\left(\sum_{k\in\mathbb{Z}}|\langle f,\psi_{j,k}\rangle|^p\right)^{1/p}b^{1/p}\\ &= C\|f\|_{\dot{B}^{1/p}_{p,1}}b^{1/p}\end{aligned}\tag{2.8}$$

since $\dfrac{1}{2}=s-\dfrac{1}{p}+\dfrac{1}{2}$ when $s=1/p$. The inequality $\|f\|_{\dot{B}^{1/p}_{p,1}}\geq cb^{-1/p}\varepsilon$ follows since $b_n\leq b$.

Further, if $C\|f\|_{\dot{B}^{1+1/p}_{p,1}}b^{1/p}\leq 1/2\|f\|_{L^p(\mathbb{R})}$ and $b_n=(a_{n+1}-a_{n-1})/2\geq b/2$ then

$$\frac{1}{2b^{1/p}}\|f\|_{L^p(\mathbb{R})}\leq\left(\sum_{n\in\mathbb{Z}}|f(a_n)|^p\right)^{1/p}\leq\frac{5}{2b^{1/p}}\|f\|_{L^p(\mathbb{R})}.$$

□

2.2. **Some corollaries.** As a first application of this proposition, let us establish the following version of the Uncertainty Principle:

Corollary 2.2. *Let $p\geq 1$ and $\alpha>0$. Let $f\in L^p$ then there exists c that depends only on p and α such that*

$$\||x|^{\alpha/p}f\|_{L^p(\mathbb{R})}\|f\|^{\alpha}_{\dot{B}^{1/p}_{p,1}}\geq c_p\|f\|^{1+\alpha}_{L^p(\mathbb{R})}.$$

Note that a stronger version has been recently obtained by J. Martin and M. Milman in [16], see the lines following inequality (5.13) in [16].

Proof. We may assume that $\|f\|_p = 1$. Suppose that $\||x|^{\alpha/p} f\|_p = A$ then for any $b > 0$

$$\int_{|x|>b/4} |f(x)|^p \, dx \leq (4b^{-1})^\alpha \int_{|x|>b/4} |x|^\alpha |f(x)|^p dx \leq 4^\alpha b^{-\alpha} A^p.$$

But

$$\int_{|x|>b/4} |f(x)|^p \, dx = \int_{-b/4}^{b/4} \sum_{j\in\mathbb{Z}\setminus\{0\}} |f(x+jb/2)|^p \, dx,$$

therefore there exists $a \in (-b/4, b/4)$ such that $\sum_{j\in\mathbb{Z}\setminus\{0\}} |f(a+jb/2)|^p \leq 4^\alpha b^{-1-\alpha} A^p$.

We choose b such that $4^\alpha b^{-\alpha} A^p = 1/2$. Then by the proposition above we obtain

$$\|f\|_{\dot{B}^{1/p}_{p,1}} \geq C b^{-1/p} = C A^{-1/\alpha}.$$

This implies the required inequality. □

We will now show some sampling estimates for functions in certain Besov spaces:

Corollary 2.3. *Let $f \in B^{1/p}_{p,1}(\mathbb{R})$ and $a = \{a_j\}$ be a sequence as above.*

(i) *If $S_1(f, a)$ is the piece-wise linear interpolant of f at a. Then*

$$\|f - S_1(f, a)\|_{L^p} \leq C b^{1/p} \|f\|_{\dot{B}^{1/p}_{p,1}}.$$

(ii) *There exists a bandlimited function $g \in [-cb^{-1}, cb^{-1}]$ such that*

$$\|f - g\|_{L^p} \leq C b^{1/p} \|f\|_{\dot{B}^{1/p}_{p,1}} \quad \text{and} \quad \|f(a_j) - g(a_j)\|_{\ell^p} \leq C \|f\|_{\dot{B}^{1/p}_{p,1}}.$$

If in addition $f \in B^s_{p,\infty}$ for some $s > 1/p$ then

$$\|f - g\|_{L^p} \leq C_s b^s \|f\|_{\dot{B}^s_{p,\infty}} \quad \text{and} \quad \|f(a_j) - g(a_j)\|_{\ell^p} \leq C_s b^{s-1/p} \|f\|_{\dot{B}^s_{p,\infty}}.$$

Proof. The first statement follows directly from the proof of the theorem. Clearly,

$$|f(x) - S_1(f, a)(x)| \leq \max\{|f(x) - f(a_{n+1})|, |f(x) - f(a_n)|\} \quad \text{for} \quad x \in [a_n, a_{n+1}].$$

Thus integrating (2.4) we obtain the required inequality.

To prove the second statement, let us now assume that ψ_1 is regular and band-limited to some interval $[-\Omega, \Omega]$. Let $f \in \dot{B}^{1/p}_{p,q}$ and write

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k} \rangle \psi_{j,k}.$$

Let j_0 be such that $2^{j_0} \leq b^{-1} \leq 2^{j_0+1}$ and define

$$g = \sum_{j\leq j_0} \sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k} \rangle \psi_{j,k} \quad \text{and} \quad h = f - g.$$

Then $\|h\|_{\dot{B}^{1/p}_{p,q}} \leq C\|f\|_{\dot{B}^{1/p}_{p,q}}$. Further we have

$$\begin{aligned}
\|h\|_{L^p} &\leq C\left(\sum_{j>j_0} 2^{(-1+p/2)j}\sum_k |\langle f,\psi_{j,k}\rangle|^p\right)^{1/p}\\
&\leq C2^{-j_0/p}\left(\sum_{j>j_1} 2^{pj/2}\sum_k |\langle f,\psi_{j,k}\rangle|^p\right)^{1/p}\\
&\leq Cb^{1/p}\sum_{j>j_0} 2^j\left(\sum_k |\langle f,\psi_{j,k}\rangle|^p\right)^{1/p}
\end{aligned}$$

and thus $\|h\|_{L^p} \leq Cb^{1/p}\|h\|_{\dot{B}^{1/p}_{p,q}} \leq Cb^{1/p}\|f\|_{\dot{B}^{1/p}_{p,q}}$.

Moreover, if we assume that $f \in B^s_{p,\infty}$ for some $s > 1/p$ then

$$\|h\|_{\dot{B}^{1/p}_{p,1}} \leq C_s b^{s-1/p}\|h\|_{\dot{B}^s_{p,\infty}} \leq C_s b^{s-1/p}\|f\|_{\dot{B}^s_{p,\infty}} \quad\text{and}\quad \|h\|_{L^p} \leq Cb^s\|f\|_{\dot{B}^s_{p,\infty}}.$$

Now, we apply (2.8) to h and obtain

$$\left(\sum_{n\in\mathbb{Z}} |h(a_n)|^p\right)^{1/p} \leq C\left(b^{-1/p}\|h\|_{L^p} + \|h\|^p_{\dot{B}^{1/p}_{p,1}}\right).$$

The required inequalities follow. □

Some results on regular smooth spline interpolation in such Besov spaces were obtained in [12], we discuss a different sampling for the multivariate case in the next section.

3. Multivariate concentration inequality and irregular sampling

3.1. **Localization inequalities in higher dimensions.** Similar argument as in the previous section gives the sampling inequalities of Theorem 1.1 formulated in the introduction for functions with small norms in appropriate homogeneous Besov spaces. Example of spaces of functions with homogeneous Besov norms controlled by the corresponding Lebesgue norm are bandlimited functions or more generally functions in shift-invariant subspaces, see [1, 2]. Sampling of bandlimited functions from irregular point sets is a well developed topic, see for example [6, 7, 8, 9, 14, 15, 19]; new interesting results on sampling from trajectories can be found in [10]. We obtain the sampling inequalities under much milder smoothness assumptions and reduce the questions of approximate sampling of functions in Besov spaces to those of bandlimited functions.

Let us now describe the general setting of our sampling sets. Our aim here is to provide a description of sampling sets that are intuitive and easy to check rather than a fully general definition that would be too complicated to check in practice. Let $1 \leq m \leq d$ be an integer and $b, C_0 > 0$ be real.

First we take G to be a finite or countable union of $d-m$ dimensional $\mathcal{C}^1$ manifolds (a countable set when $m = d$). To each $a \in G$ we attach an m-dimensional manifold H_a in a sufficiently regular way (*e.g.* H_a may be defined through an implicit function). Each H_a is endowed with a measure ν_a that is absolutely continuous with respect to the corresponding Hausdorff (surface) measure on H_a, $\nu_a = \varphi \mathrm{d}\mathcal{H}^m\Big|_{H_a}$ with $C_0^{-1} \leq \varphi \leq C_0$, measures ν_a depend on a also in a regular way, see (ii) below. We further assume that

(i) for any a the diameter of H_a is bounded in the following way $b/2 \leq \operatorname{diam}(H_a) \leq b$,

(ii) for every measurable set $E \subset \mathbb{R}^d$ sets $E \cap H_a$ are ν_a-measurable for $\mathcal{H}^{m-d}$-almost all $a \in G$ and $C_0^{-1}\lambda_d(E) \le \int_G \nu_a(E \cap H_a) d\mathcal{H}^{m-d}(a) \le C_0 \lambda_d(E)$; this implies that for every $f \in L^1(\mathbb{R}^d)$,

$$\frac{1}{C_0}\int_{\mathbb{R}^d} |f(x)|\,\mathrm{d}x \le \int_G \int_{H_a} |f(x)|\,\mathrm{d}\nu_a(x)\,\mathrm{d}\mathcal{H}^{d-m}(a) \le C_0 \int_{\mathbb{R}^d} |f(x)|\,\mathrm{d}x, \tag{3.9}$$

(iii) for every $x \in \mathbb{R}^d$, $R > 0$,

$$C_0^{-1}\min(R,b)^m \le \nu_a\big(B(x,R) \cap H_a\big) \le C_0 \min(R,b)^m, \tag{3.10}$$

(iv) for every $x \in \mathbb{R}^d$, $R > 0$

$$\mathcal{H}^{d-m}(G \cap B(x,R)) \le C_0 R^{d-m} \max\{1, Rb^{-1}\}^m. \tag{3.11}$$

Example 3.1. (i) Let $(a_n)_{n\in\mathbb{Z}}$ be an increasing sequence such that $\frac{b}{2} \le a_{n+1} - a_n \le b$ then we can take $G = \cup_{n\in\mathbb{Z}} P_n$, where $P_n = \mathbb{R}^{d-1} \times \{a_n\}$, here $m = 1$. For $a = (\alpha, a_n) \in G$ we take $H_a = \{\alpha\} \times \left[\dfrac{a_{n-1}+a_n}{2}, \dfrac{a_{n+1}+a_n}{2}\right]$.

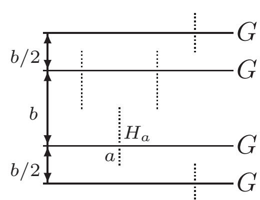

Figure 1. Example (i) in dimension 2, $G = \bigcup \mathbb{R} \times \{a_n\}$

(ii) The hyperplanes P_n can be replaced by more general manifolds. For instance, let $(a_n)_{n\in\mathbb{Z}}$ be a sequences such that $\frac{b}{2} \le a_{n+1} - a_n \le b$ and let $f : \mathbb{R}^{d-1} \to \mathbb{R}$ be smooth bounded function with bounded derivative. Let $\tilde{P}_n = \big\{\big(x', f(x') + a_n\big), x' \in \mathbb{R}^{d-1}\big\}$ then we can take $G = \cup_{n\in\mathbb{Z}} \tilde{P}_n$. Then if $a = \big(x', f(x') + a_n\big)$ we can again set $H_a = \{a + (0,t) : |t| \le b + \max|f|\}$.

(iii) For each $k \in \mathbb{Z}^{d-1}$, let $f_k : \mathbb{R} \to \mathbb{R}^{d-1}$ be a smooth function such that f_k takes its values in the ℓ^∞-ball centered at bk of radius $b/4$ and such that the derivatives are uniformly bounded from above and below $C_0^{-1} \le \|f_k'\| \le C_0$. Let $P_k = \big\{\big(f_k(x), x\big), x \in \mathbb{R}\big\} \subset \mathbb{R}^d$ and $G = \bigcup_{k\in\mathbb{Z}^{d-1}} P_k$. To each $\big(f_k(a), a\big) \in G$ we associate H_a to be the ℓ^∞-ball centered at (bk, a) of radius $b/4$ endowed with the Lebesgue measure.

We will prove the following generalization of Theorem 1.1

Theorem 3.2. *Suppose that $1 \le p \le \infty$ and G as above. There exist constants δ, C_1, C_2 that depend only on d, p, m, D, C_0 such that if $f \in B^{m/p}_{p,1}(\mathbb{R}^d)$, $\|f\|_{\dot{B}^{m/p}_{p,1}} \le N\|f\|_p$, and $b^{m/p} < \delta N^{-1}$ then*

$$C_1\|f\|_{L^p} \le b^{m/p}\left(\int_G |f(x)|^p d\mathcal{H}^{d-m}(x)\right)^{1/p} \le C_2\|f\|_{L^p}.$$

Proof. The proof repeats the one-dimensional argument from the previous section. We start with a smooth one-dimensional wavelet function supported on $[-R, R]$ and construct

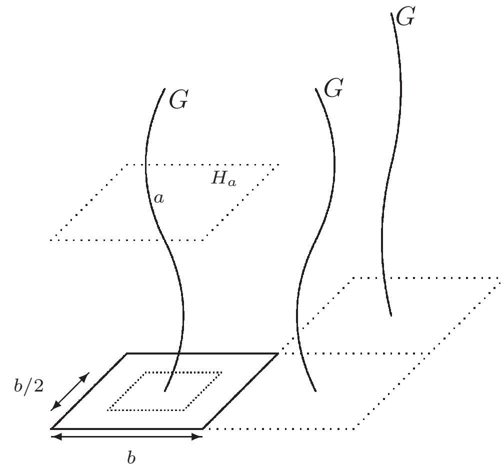

FIGURE 2. Example (iii) in dimension 3

a wavelet basis as outlined in the introduction. We have

$$f = \sum_{j\in\mathbb{Z}} \sum_{\lambda\in\mathbb{Z}^d\times L^d} \langle f, \psi_{j,\lambda}\rangle \psi_{j,\lambda}.$$

Fix some a and let $x \in H_a$. Then

$$f(x) - f(a) = \sum_{j\in\mathbb{Z}} 2^{dj/2} \sum_{\lambda=(k,l)\in\mathbb{Z}^d\times L^d} \langle f, \psi_{j,\lambda}\rangle \big(\psi_{0,l}(2^j x - k) - \psi_{0,l}(2^j a - k)\big).$$

Write

$$Z_x(j) = \{k \in \mathbb{Z}^d : 2^j x - k \in [-R, R]^d\},\ Z_x(a, j) = Z_x(j) \cup Z_a(j),$$
$$Z(a, j) = \cup_{x\in H_a} Z_x(j),$$

and note that, $|Z_x(j)| \le C$, where C depends on R only. We also define

$$E_{a,j} = \left(\sum_{\lambda\in Z(a,j)\times L^d} |\langle f, \psi_{j,\lambda}\rangle|^p\right)^{1/p}.$$

Further, it is not difficult to see that $|Z(a, j)| \le C(1 + 2^{jm} b^m)$ and by (3.11) the following inequality holds

$$M_j := \sup_{k\in\mathbb{Z}^d} \mathcal{H}^{d-m}\{a \in G : k \in Z(a, j)\}| \le \begin{cases} C_0 2^{-j(d-m)} & \text{if } j \ge j_0 \\ C_0 b^{-m} 2^{-dj} & \text{if } j < j_0 \end{cases},$$

where $2^{-j_0} \in (b, 2b]$ as before. Then (2.7) becomes

$$|f(x) - f(a)| \le C \sum_{j\ge j_0} 2^{dj/2} E_{a,j} + C \sum_{j<j_0} 2^{(d+2)j/2} |x - a| E_{a,j}.$$

We take the $L^p(H_a,\nu_a)$-norms over H_a. Then we apply the triangle inequality and get

$$\left|\left(\int_{H_a}|f(x)|^p d\nu_a\right)^{1/p} - |f(a)|\nu_a(H_a)^{1/p}\right| \\ \leq C\sum_{j\geq j_0} 2^{dj/2}b^{m/p}E_{a,j} + C\sum_{j<j_0} 2^{(d+2)j/2}b^{1+m/p}E_{a,j}.$$

Next, taking the L^p norm over each G, we have

$$\left|\left(\int_G\int_{H_a}|f(x)|^p\,\mathrm{d}\nu_a(x)\,\mathrm{d}\mathcal{H}^{d-m}(a)\right)^{1/p} - \|f(a)\nu_a(H_a)^{1/p}\|_{L^p(G)}\right| \\ \leq C\sum_{j\geq j_0} 2^{dj/2}b^{m/p}\|E_{a,j}\|_{L^p(G)} + C\sum_{j<j_0} 2^{(d+2)j/2}b^{1+m/p}\|E_{a,j}\|_{L^p(G)}. \tag{3.12}$$

Futher, when $j\geq j_0$ the estimate for M_j above implies

$$\|E_{a,j}\|_{L^p(G)} \leq C_0 2^{-j(d-m)/p}\left(\sum_{\lambda=(k,l)}|\langle f,\psi_{j,\lambda}\rangle|^p\right)^{1/p}.$$

Then the first sum is the right hand side of (3.12) is bounded by $Cb^{m/p}\|f\|_{\dot{B}^{m/p}_{p,1}}$.

For $j<j_0$, we have

$$\|E_{a,j}\|_{L^p(G)} \leq C_0 b^{-m/p}2^{-jd/p}\left(\sum_{\lambda=(k,l)}|\langle f,\psi_{j,\lambda}\rangle|^p\right)^{1/p}.$$

We divide the second sum in the right hand side of (3.12) into two sums, where $j_1<j_0$ will be fixed later,

$$\begin{aligned} C\sum_{j<j_0} 2^{(d+2)j/2}b^{1+m/p}\|E_{a,j}\|_{L^p(G)} &\leq C\left(\sum_{j<j_1}+\sum_{j=j_1}^{j_0}\right) b2^{j(d/2+1-d/p)}\left(\sum_{\lambda=(k,l)}|\langle f,\psi_{j,\lambda}\rangle|^p\right)^{1/p} \\ &\leq Cb\sum_{j<j_1}2^j\|f\|_{L^p(\mathbb{R}^d)} + Cb\max_{j_1\leq j<j_0}\{2^{j(1-m/p)}\}\|f\|_{\dot{B}^{m/p}_{p,1}} \\ &\leq Cb2^{j_1}\|f\|_{L^p(\mathbb{R}^d)} + Cb\left(2^{j_0(1-m/p)}+2^{j_1(1-m/p)}\right)\|f\|_{\dot{B}^{m/p}_{p,1}}, \end{aligned}$$

where the constants depend on ψ and does not depend on j (it follows by a simple scaling argument). Since $b2^{j_0}\in[1/2,1]$, we obtain

$$\sum_{j<j_0} 2^{(d+2)j/2}b^{1+m/p}\|E_{a,j}\|_{L^p(G)} \leq \\ C2^{j_1-j_0}\|f\|_{L^p(\mathbb{R}^d)} + Cb^{m/p}2^{m/p}\left(1+2^{(j_1-j_0)(1-m/p)}\right)\|f\|_{\dot{B}^{m/p}_{p,1}}.$$

In summary, (3.12) implies

$$\left|\left(\int_G\int_{H_a}|f(x)|^p\,\mathrm{d}\nu_a(x)\,\mathrm{d}\mu(a)\right)^{1/p} - \|f(a)\nu_a(H_a)^{1/p}\|_{L^p(G)}\right| \\ \leq C2^{j_1-j_0}\|f\|_{L^p(\mathbb{R}^d)} + Cb^{m/p}C_{m,p}\|f\|_{\dot{B}^{m/p}_{p,1}}, \tag{3.13}$$

where $C_{m,p}=C\Big(1+2^{m/p}\big(1+2^{(j_1-j_0)(1-m/p)}\big)\Big)$. Taking the lower bound in (3.9) and the upper bound in (3.10) we get

$$C_0^{-1}\|f\|_{L^p(\mathbb{R}^d)}-C_0^{1/p}b^{m/p}\|f\|_{L^p(G)}\leq C2^{j_1-j_0}\|f\|_{L^p}+Cb^{m/p}C_{m,p}\|f\|_{\dot B^{m/p}_{p,1}}.$$

Choosing j_1 small enough to have $C2^{j_1-j_0}\leq(2C_0)^{-1}$ we get

$$(2C_0)^{-1}\|f\|_{L^p(\mathbb{R}^d)}-C_{m,p}b^{m/p}\|f\|_{\dot B^{m/p}_{p,1}}\leq C_0^{1/p}b^{m/p}\|f\|_{L^p(G)}.$$

On the other hand, taking the upper bound in (3.9) and the lower bound in (3.10), (3.13) with $j_1=j_0$ implies

$$C_0^{-1/p}b^{m/p}\|f\|_{L^p(G)}-C_0\|f\|_{L^p(\mathbb{R}^d)}\leq C\|f\|_p+C(1+2^{m/p+1})b^{m/p}\|f\|_{\dot B^{m/p}_{p,1}}.$$

The theorem follows immediately. □

Remark. The result may be extended to sampling sets that do not exactly satisfy the requirements. For instance let $\{r_n\}_{n\in\mathbb{N}}$ be an increasing sequence of positive numbers, $r_{n+1}-r_n\in(b/2,b)$ and let $G=\cup_n\{x:|x|=r_n\}$ be the union of concentric spheres of radius r_n. For $a\in G$, $|a|=r_n$, we define

$$H_a=\begin{cases}\left\{ta,\ \frac{r_{n-1}+r_n}{2}\leq t\leq\frac{r_{n+1}+r_n}{2}\right\}&\text{for } n\geq1\\ \left\{ta,\ 0\leq t\leq\frac{r_0+r_1}{2}\right\}&\text{for } n=0\end{cases}.$$

The right hand side of (3.9) does not hold and we have to replace it by

$$\int_G\int_{H_a}|f(x)|\,\mathrm{d}\nu_a(x)\,\mathrm{d}\mathcal{H}^{d-m}(a)\leq C_0\int_0^\infty\int_{\mathbb{S}^{d-1}}|f(r\zeta)|\,\mathrm{d}\sigma(\zeta)\max(1,r^{d-1})\,\mathrm{d}r \tag{3.14}$$

while the left hand side of (3.9) still holds.

Then the proof of the theorem shows that

$$C_1\|f\|_{L^p(\mathbb{R}^d)}\leq b^{m/p}\left(\int_G|f(x)|^p d\mathcal{H}^{d-m}(x)\right)^{1/p}\leq C_2\|f\|_{L^p(\mathbb{R}^d,\nu)}$$

where ν is the measure $\mathrm{d}\sigma(\zeta)\max(1,r^{d-1})\,\mathrm{d}r$ (in polar coordinates). We then apply the left hand side of this inequality to $f\alpha$ where α is a smooth function such that $\alpha=0$ in a ball $B(0,\varepsilon)$ and $\alpha=1$ outside the ball $B(0,1/4)$. Then $\|\alpha f\|_{B^{m/p}_{p,1}}\leq C\|f\|_{B^{m/p}_{p,1}}$ so that the theorem applies if b is small enough. It remains to notice that

$$\begin{aligned}\left(\int_G|f(x)|^p d\mathcal{H}^{d-m}(x)\right)^{1/p}&=\left(\int_G|\alpha(x)f(x)|^p d\mathcal{H}^{d-m}(x)\right)^{1/p}\\&\leq C_2\|\psi f\|_{L^p(\mathbb{R}^d,\nu)}\leq C\|\alpha f\|_{L^p(\mathbb{R}^d)}\leq C\|f\|_{L^p(\mathbb{R}^d)}.\end{aligned}$$

Therefore the theorem also holds in this case.

A similar reasoning also applies to a spiral. Let r_k be as previously and let ρ be a smooth (strictly) increasing function such that $\rho(2k\pi)=r_k$. Let $G\subset\mathbb{R}^2$ be the curve given in polar coordinates by ρ, that is $G=\{\rho(\theta)(\cos\theta,\sin\theta),\theta\in[0,+\infty)\}$. If $\theta\in[2k\pi,2(k+1)\pi]$ we attach to $a=\rho(\theta)(\cos\theta,\sin\theta)$ the manifold $H_a=t(\cos\theta,\sin\theta)$, $r_{k-1}\leq t\leq r_{k+1}$ (with the convention $r_{-1}=0$).

3.2. **Irregular sampling.** Let us now outline how the above results fit into existing general sampling procedures. Suppose that $f \in B^{m/p}_{p,1}(\mathbb{R}^d)$ and G is the union of $d-m$-dimensional subspaces as in the theorem, then we have the trace operator bounded in the following way

$$T_G : B^{m/p}_{p,1}(\mathbb{R}^d) \to L^p(G), \quad \|T_G f\|_{L^p(G)} \leq C\left(b^{-m/p}\|f\|_{L^p(\mathbb{R}^d)} + \|f\|_{B^{m/p}_{p,1}(\mathbb{R}^d)}\right).$$

Assume further that we are given a bounded operator $S_G : L^p(G) \to L^p(\mathbb{R}^d)$ that interpolates band-limited functions. More precisely, we assume that there is an $A > 0$ such that

$$\|S_G u\|_{L^p(\mathbb{R}^d)} \leq Ab^{m/p}\|u\|_{L^p(G)}$$

and if g is bandlimited with $\hat{g}(x) = 0$ when $x \notin [-cb^{-1}, cb^{-1}]^d$ then

$$\|g - S_G(T_G g)\|_{L^p(\mathbb{R}^d)} \leq Bb^{m/p}\|g\|_{\dot{B}^{m/p}_{p,1}}.$$

Now, let $f \in B^{m/p}_{p,1}$. Fix a smooth bounded multiplier χ, such that $\operatorname{supp}(\chi) \subset [-cb, cb]^d$ and $\chi = 1$ on $[-ab^{-1}, ab^{-1}]^d$. Write $f = g + h$, where $g = P_\chi f = \mathcal{F}^{-1}(\hat{f}\chi)$ so that $\operatorname{supp}(g) \subset [-cb, cb]^d$ and $\hat{h} = 0$ on $[-ab^{-1}, ab^{-1}]^d$. Then

$$\begin{aligned} \|f - S_G(T_G f)\|_{L^p(\mathbb{R}^d)} &\leq \|h\|_{L^p} + \|g - S_G(T_G g)\|_{L^p(\mathbb{R}^d)} + \|S_G(T_G h)\|_{L^p(\mathbb{R}^d)} \\ &\leq \|g - S_G(T_G g)\|_{L^p(\mathbb{R}^d)} + \|h\|_{L^p(\mathbb{R}^d)} + Ab^{m/p}\|h\|_{L^p(G)}. \end{aligned}$$

Applying the theorem we see that the last two terms are bounded by $C(1+A)b^{m/p}\|f\|_{B^{m/p}_{p,1}}$. If we further assume better smoothness for f, $f \in B^s_{p,\infty}$ with $s > 1/p$, then they are even bounded by $C_s(1+A)b^s\|f\|_{B^s_{p,\infty}}$. Thus sampling of functions in Besov spaces can be reduced to sampling of bandlimited functions. This was done in [12] for the case of dimension one and regular samples. We allow irregular sample sets and to claim the correct order of converges of reconstructions,

$$\|f - S_G(T_G f)\|_p \leq Cb^{m/p}\|f\|_{B^{m/p}_{p,1}},$$

we need the constants A and B be uniform, i.e., they may depend only on p, m, d and D but not an the specific geometry of the set G.

It can be checked that for example the (iterative) sampling algorithm provided in [1] can be applied, where the L^2-estimates can be replaced by L^p-estimates, related inequalities can be found in [7, 20]. Let $G = \Gamma \times \mathbb{R}^m$ as above, and let $\Lambda = (b\mathbb{Z})^m$. We consider $\Lambda_G = \Gamma \times \Lambda$, it is a discrete separated set in $\mathbb{R}^d$. The usual averaging operator $V : L^p(G) \to l^p(\Lambda_G)$ is bounded, $\|V(u)\|_{l^p(\Lambda_G)} \leq b^{(m-d)/p}\|u\|_{L^p(G)}$. Let $\{\beta_j\}$ be a smooth bounded partition of unity adapted to $\{B(x, 2b)\}_{x \in \Lambda_g}$ (see for example Definition 4.2 in [1]). We define $A : l^p(\Lambda_G) \to L^p$ by $Ac = \sum_j c_j\beta_j$ and $A_1 = AVT_G$. Then by Lemma 4.1 in [6] choosing c small enough we can guarantee that $I - P_\chi A_1$ is a contraction on $L^p \cap \mathcal{F}^{-1}(L^2([-cb, cb]^d)$, where $P_\chi : L^p \to L^p$ is defined by $P_\chi f = (\mathcal{F})^{-1}(\chi\hat{f})$ as above. Then there exists N depending only on p, m, d, c such that

$$S_G = \sum_{k=0}^{N}(I - PA_1)^k P_\chi AV_G$$

satisfies the required uniform estimates.

References

[1] A. Aldroubi, *Non-uniform weighted average sampling and reconstruction in shift-invariant and wavelet spaces*, Appl. Comput. Harmon. Anal. **13** (2002), 151–161.

[2] A. Aldroubi & K. Gröchenig *Nonuniform sampling and reconstruction in shift-invariant spaces*, SIAM Rev. **43** (2001), 585–620

[3] G. Bourdaud, *Ondelettes et espaces de Sobolev*, Rev. Mat. Iberoam. **11** (1995), 477–512.

[4] M. G. Cowling & J. F. Price, *Bandwidth versus time concentration: the Heisenberg-Pauli-Weyl inequality*, SIAM J. Math. Anal. **15** (1984), 151–165.

[5] I. Daubechies, *Ten Lectures on Wavelets*, CBMS-NSF regional conference series in applied mathematics **61**. SIAM, 1992

[6] H. Feichtinger & K. Gröchenig, *Iterative reconstruction of multivariate band-limited functions from irregular sampling values*, SIAM J. Math. Anal. **23** (1992), no. 1, 244–261.

[7] H. Feichtinger & K. Gröchenig, *Irregular sampling theorems and series expansion of band-limited functions*, J. Math. Anal. Appl., **167** (1992), 530–556.

[8] K. Gröchenig *Reconstruction algorithms in irregular sampling.* Math. Comp. **59** (1992), 181–194.

[9] K. Gröchenig *Irregular sampling of wavelet and short-time Fourier transforms.* Constr. Approx. **9** (1993), 283–297.

[10] K. Gröchenig, J. L. Romero, J. Unnikrishnan & M. Vetterli, *On minimal trajectories for mobile sampling of bandlimited fields*, arXiv:1312.7794

[11] St. Jaffard & Y. Meyer *On the regularity of functions in critical Besov spaces*, J. Funct. Anal., **175** (2000), 415–434.

[12] St. Jaffard, M. Okada & T. Ueno *Approximate sampling theorem and the order of smoothness of the Besov space,* Harmonic analysis and nonlinear partial differential equations, 45–56, RIMS Kôkyûroku Bessatsu, B18, Res. Inst. Math. Sci. (RIMS), Kyoto, 2010.

[13] G. Kyriazis *Wavelet coefficients measuring smoothness in $H^p(\mathbb{R}^d)$*, Appl. Comp. Har. Anal. **3** (1996), 100–119.

[14] Yu. Lyubarskii & W. R. Madych, *The recovery of irregular sampled bond limited functions via tempered splines*, J. Funct. Anal., **125** (1995), 201–222.

[15] W. R. Madych, *An estimate for multivariate interpolation, II*, J. Approx. Theory, **142** (2006), 116–128.

[16] J. Martin & M. Milman, *Isoperimetric weights and generalized uncertainty inequalities in metric measure spaces.* arXiv:1501.06556 [math.FA]

[17] Y. Meyer, *Ondelettes et opérateurs*, Hermann, Paris 1990

[18] J. Peetre, *New thoughts on Besov spaces*, Duke Univ. Math. Series, Durham, 1976.

[19] I. Pesenson, *A reconstruction formula for band limited functions in $L^2(\mathbb{R}^d)$*, Proc. AMS, **127** (1999), 3593–3600.

[20] I. Pesenson, *Plancherel-Polya-type inequalities for entire functions of exponential type in $L_p(\mathbb{R}^d)$*, J. Math. Anal. Appl, **330** (2007), 1194–1206.

[21] R. S. Strichartz, *Uncertainty principles in harmonic analysis*, J. Funct. Anal., **84** (1989), 97–114.

[22] H. Triebel, *Characterizations of Besov-Hardy-Sobolev spaces via harmonic functions, temperatures, and related means*, J. Approx. Theory **35** (1982), 275–297

[23] H. Triebel, *Theory of function spaces III*, Monographs in Math., vol. 100, Birkhaüser Verlag, 2006.

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